CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Example Exam Wiskunde A

Date: Autumn 2018
Time: 3 Hours
Questions: 6

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in reduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put in your bag.

Points that can be scored for each question:

Question	1	2	3	4	5	6
a	4	4	4	5	4	3
b	4	4	2	4	4	3
c	4	4	3		4	5
d	5	6	3		5	3
e					3	
Total	17	18	12	9	20	14

$$\text{Grade} = \frac{\text{total points scored}}{10} + 1$$

You will pass the exam if your grade is at least 5.5 .

Question 1 – Algebraic computations

Source: CCVX entrance exam wiskunde A July 2018

When you are asked to perform a computation ***algebraically****, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed as well in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like* $\sqrt{2}$ *and* $\log(3)$*.*

Solve the following equations algebraically. If there is a square root or a logarithm in your answer, give the answer rounded to three digits behind the decimal point.

4pt a $9x^3 + 3x^2 = 2x$

4pt b $3 \cdot 5^x + 14 = 10 \cdot 5^x$

The function f is given by

$$f(x) = \frac{x^2}{x+2}$$

4pt c Compute algebraically the slope of the graph of f in point $\left(1, \frac{1}{3}\right)$.

5pt d Compute algebraically the coordinates of the intersection(s) of the graph of f and the line $y = \frac{3}{4}x - \frac{1}{2}$.

Question 2 – Maximizing profit

Source: CCVX entrance exam wiskunde A July 2018

Bert produces and sells unique souvenirs of the Dom-tower in Utrecht. The number of souvenirs that he produces per day depends solely on his investment for producing these souvenirs. This number is given by

$$Q(I) = \sqrt{5I - 10}$$

with I the investment in euros.

The price at which all souvenirs will be sold depends on the number of souvenirs he produces. This price (in euros) is given by

$$P(Q) = 20 - \tfrac{1}{5}Q$$

Bert decides to use this formula to determine the selling price.
The revenue of the sale of these souvenirs in euro's is then given by

$$R = QP = Q(20 - \tfrac{1}{5}Q)$$

4pt **a** Use the derivative $\frac{\mathrm{d}R}{\mathrm{d}Q}$ to determine algebraically the number of produced souvenirs at which the revenue is maximal.

4pt **b** Compute algebraically Bert's investment if the selling price of the souvenirs is 12 euro.

The formulas given above can be combined into one formula which yields Bert's profit (in euros) as a function of his investment (in euros). This function is given by the formula

$$Profit = 2 - 2I + 20\sqrt{5I - 10}$$

4pt **c** Show that this formula is correct by substituting the formulas given above into the formula $Profit = Revenue - Investment$.

6pt **d** Compute the maximal profit algebraically.

Question 3 – Tonic

Source: CCVX entrance exam wiskunde A December 2017 (adjusted)

The company Royal Pub produces litre bottles of tonic. The volume of tonic in these bottles is normally distributed with average of $\mu_V = 1040$ ml and standard deviation $\sigma_V = 23$ ml.

The density of the tonic is 1.087 g/ml . This means that the weight of the tonic in these bottles is normally distributed with mean $\mu_G = 1130.5$ g and standard deviation $\sigma_G = 25.0$ g.

The bottles are made of plastic. The weight of empty bottles is normally distributed with mean $\mu_F = 90.5$ g and standard deviation $\sigma_F = 3.0$ g.

4pt **a** Compute the mean and the standard deviation of the total weight of a bottle that is filled with tonic.

The company regularly checks whether the mean volume of tonic in the bottles of tonic is still 1040 ml. This is tested by determining the average volume of tonic in a sample of 100 bottles. In this test, it is assumed that the standard deviation is still 23 ml and that the significance level is $\alpha = 5\%$.

2pt **b** Formulate the null hypothesis and the alternative hypothesis for this testing procedure.

The average volume of tonic in the 100 bottles in the sample turns out to be 1036 ml. This yields a p-value of 0.041.

3pt **c** Find the parameters of the test statistic that is used to compute this p-value.

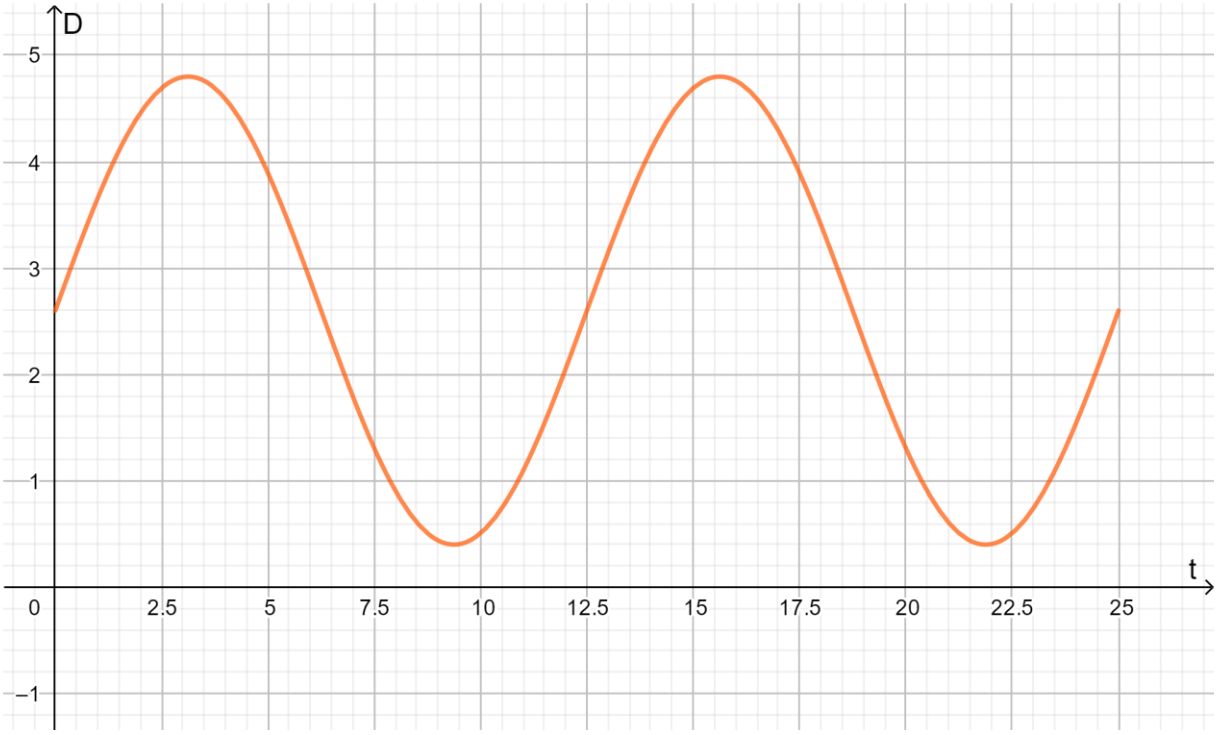
3pt **d** What is the conclusion of the testing procedure with this outcome of the sample? Give a motivation for your answer!

Question 4 – Tides in a harbour

Source: CCVX entrance exam wiskunde A July 2018

Tides are the rise and fall of sea levels caused by the combined effects of the gravitational forces exerted by the Moon and the Sun and the rotation of Earth.

As a consequence of this phenomenon, the depth of the harbour of Sinustown varies over time. For the first 25 hours of August 2018, *D* the depth of this harbour in meters, as a function of *t*, the time in hours, is depicted in the figure below.



The function of which the graph is depicted in this figure has a formula of the form

$$D(t) = A + B\sin(Ct)$$

5pt a Find values of *A, B* and *C* which match the description given above.

On midnight in the night from 1 August to 2 August, the depth of the harbour is 1.54 m.

4pt b Assuming that the graph continues similarly for the next day, algebraically find the times on 2 August at which the depth of the harbour also equals 1.54 m.

Question 5 – Three growth models

Source: CCVX entrance exam wiskunde A July 2018 (adjusted)

On a large university campus, three students return together from a trip abroad carrying a highly contagious but otherwise harmless disease. Someone who is affected by this disease, suffers some minor discomforts after a few days, but will not have any symptoms afterwards. However, he or she will be a carrier of this disease for the rest of his or her life.
Two days after the return of the three students, 48 students are carrying this disease and four days after their return, 768 students are carrying this disease.

4pt **a** Is there a **linear** growth model that fits these numbers?
If not, explain why; if so, calculate the number of students carrying this disease according to this model three days after the return of the three students.

4pt **b** Is there an **exponential** growth model that fits these numbers?
If not, explain why; if so, calculate the number of students carrying this disease according to this model three days after the return of the three students.

With both a linear and an exponential growth model, the number of students carrying this disease will rise above each limit in the long run. That is why a third growth model is considered. According to this third growth model, from the fourth day after the return of the three students the number of students carrying this disease is approximately given by

$$N(t) = 5000 \cdot (3 - 4e^{-0.085t})$$

In this formula, t is the time in days with $t = 0$ on the day the three students return to the campus, and $N(t)$ is the number of students carrying this disease.

4pt **c** Compute the rate of growth of the number of students carrying this disease according to this third model at $t = 4$. (rate of growth = derivative of the growth function.)

5pt **d** Compute algebraically the time at which 10 000 students will be carrying this disease according to this formula.
Give the answer rounded to whole hours.

There are 20 000 students living on the campus.

3pt **e** How many of these students will **not** be carrying this disease in the long run according to this formula?

Question 6 – Holiday bungalows

Source: CCVX entrance exam wiskunde A July 2018 (adjusted)

RJ owns a small holiday resort with 18 bungalows. In the last week of July, these are in such high demand that he will rent out all bungalows on offer. However, from experience he knows that 10% of the tenants do not show up. Therefore, he considers to rent out 20 bungalows. Then, if 19 or 20 tenants show up, he has to find other accommodation for one or two tenants and their families.

In this question we assume that tenants show up independently of each other.

3pt **a** Compute the probability that exactly 18 tenants turn up if he rents out 20 bungalows.

3pt **b** Show that the probability that 19 out of 20 tenants show op is 0.2702 and that the probability that all 20 tenants show up is 0.1216 (rounded to four decimal places).

Each tenant has to pay the full amount of 2000 euros per week for the rent of the bungalow in advance. This means that RJ has a revenue of 40 000 euros if he rents out 20 bungalows. However, for each tenant that he cannot accommodate in his 18 bungalows, he has to pay 4000 euros for alternative accommodation.

5pt **c** Find out which is expected to be more profitable: renting out 18 bungalows and thus missing extra income if at least two tenants do not show op, or renting out 20 bungalows with the risk of having to pay for alternative accommodation if 19 or 20 tenants show up.

Another possibility is of course to rent out 19 bungalows.

3pt **d** Compute RJ's expected net income for this possibility.

End of the exam.

Is your name on all answer sheets?

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	function	derivative function
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\frac{dk}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Logarithms

Rule	conditions
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \frac{a}{b}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0, \quad g \neq 1, \quad a > 0$
${}^{g}\log a = \frac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad p > 0, \quad p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \frac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* =number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If *X* and *Y* are any random variables, then: $E(X+Y)=E(X)+E(Y)$
If furthermore *X* and *Y* are independent, then: $\sigma(X+Y)=\sqrt{\sigma^2(X)+\sigma^2(Y)}$

$\sqrt{n}$-law:
For *n* independent repetitions of the same experiment where the result of each experiment is a random variable *X*, the sum of the results is a random variable *S* and the mean of the results is a random variable $\overline{X}$.

$E(S)=n\cdot E(X)$ $\qquad$ $\sigma(S)=\sqrt{n}\cdot\sigma(X)$

$E(\overline{X})=E(X)$ $\qquad$ $\sigma(\overline{X})=\dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If *X* has a binomial distribution with parameters *n* (number of experiments) and *p* (probability of success at each experiment), then

$$P(X=k)=\binom{n}{k}\cdot p^k\cdot(1-p)^{n-k} \quad \text{with} \quad k=0,1,2,\ldots,n$$

Expected value: $E(X)=np$ $\qquad$ Standard deviation: $\sigma(X)=\sqrt{n\cdot p\cdot(1-p)}$

n and *p* are the parameters of the binomial distribution

Normal Distribution

If *X* is a normally distributed random variable with mean μ and standard deviation σ, then

$Z=\frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X<g)=P\left(Z<\frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic *T* is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are::

α	left sided	right sided	two sided
0.05	$g=\mu_T-1.645\sigma_T$	$g=\mu_T+1.645\sigma_T$	$g_l=\mu_T-1.96\sigma_T$
			$g_r=\mu_T+1.96\sigma_T$
0.01	$g=\mu_T-2.33\sigma_T$	$g=\mu_T+2.33\sigma_T$	$g_l=\mu_T-2.58\sigma_T$
			$g_r=\mu_T+2.58\sigma_T$

Extra question – A hard exam?

Source: CCVX entrance exam wiskunde A July 2018 (adjusted)

An exam consists of 10 multiple choice questions and 4 open questions. The time that well prepared candidates need to answer a multiple choice question is normally distributed with a mean of 5 minutes and a standard deviation of 1 minute. The time that they need to answer an open question is normally distributed with a mean of 15 minutes and a standard deviation of 3 minutes. In this question we assume that the times that a candidate needs to answer these questions are independent of each other.

The total time that a well prepared candidate needs to finish all questions is a normally distributed random variable X.

5pt a Compute the mean and the standard deviation of X.

3pt b According to the rules of thumb, what percentage of these candidates will need more than 6 minutes to answer a multiple choice question?

After the exam, candidates complain that this exam is much harder than last year's exam. The exam board decides to test this by checking the exams of 50 candidates and comparing the pass rate of these candidates to last year's pass rate, which was 80%. They thereby take a significance level of $\alpha = 5\%$.

2pt c Formulate the null hypothesis and the alternative hypothesis for this testing procedure.

5pt d What is the outcome of this test if 36 of these 50 candidates pass the exam?