CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Entrance Exam Wiskunde A

Date: 19 December 2018
Time: 13.30 – 16.30 hours
Questions: 6

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in reduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put in your bag.

Points that can be scored for each question:						
Question	1	2	3	4	5	6
a	4	2	5	6	3	5
b	4	2	5	5	4	2
c	5	4	4	4	4	3
d	5	2	4	3	3	2
Total	18	10	18	18	14	12

$$\text{Grade} = \frac{\text{total points scored}}{10} + 1$$

You will pass the exam if your grade is at least 5.5 .

Question 1 – Algebraic computations

When you are asked to perform a computation ***algebraically****, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed as well in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like* $\sqrt{2}$ *and* $\log(3)$*.*

Solve the following equations algebraically. If there is a square root or a logarithm in your answer, give the answer rounded to three digits behind the decimal point.

4pt a $9x^3 + 25x = 30x^2$

4pt b $5 \cdot 4^x = 2 \cdot 5^x$

Given are the function f with function rule $f(x) = \sqrt{8x - 12}$ and the line l with equation $y = 2x - 6$.

5pt c Use the derivative function of f to show that the tangent line to the graph of f in point $A(2,2)$ is parallel to line l.

5pt d Compute algebraically the coordinates of the intersection(s) of the graph of f and line l.

Question 2 – Born on the waves

The yearly number of births in country C has been stable for years, but the number of births per week does vary throughout the year. According to the Statistical Office of country C, this number is given by the formula

$$B(t) = 5200 + 200\sin\left(\frac{\pi}{13}\left(t - 8\tfrac{1}{2}\right)\right)$$

In this formula, *B* is the number of births per week, and *t* is the time in weeks, with $t = 0$ on 1 January.

2pt a Compute the period of the function *B* algebraically.

2pt b Compute the maximal value of *B* algebraically.

4pt c Compute algebraically all times between $t = 0$ and $t = 52$ at which the number of births per week according to this formula is equal to 5000.

2pt d Compute the total number of births in one year (= 52 weeks) according to this formula.

Question 3 – A drug

A drug is injected into the bloodstream of a patient. The concentration of this drug in the blood flowing through the lungs is, in a first model, given by

$$C_1(t) = \frac{13t}{t^2 + 4}$$

(C_1 in milligrams per liter, t is the time in hours after administration of the drug).

In figure 1 a sketch of the graph of C_1 is shown.

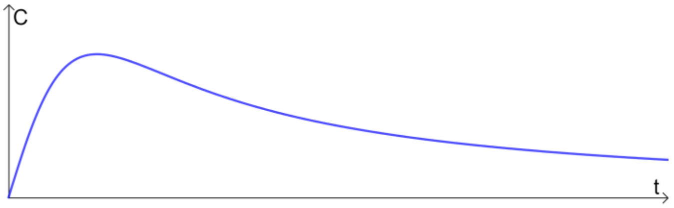

Figure 1

The drug is effective if the concentration is more than 1.25 milligrams per liter.

5pt a Solve the equation $C_1(t) = 1.25$ algebraically and compute how long the drug is effective. Give your answer in minutes.

5pt b Compute the maximal value of $C_1(t)$ algebraically.

According to a second model, the concentration of the drug in the blood flowing through the lungs is given by the formula

$$C_2(t) = 4.5t \cdot \mathrm{e}^{-0.5t}$$

with C_2 in milligrams per liter and t as above.
The graph of this function is shown in figure 2.

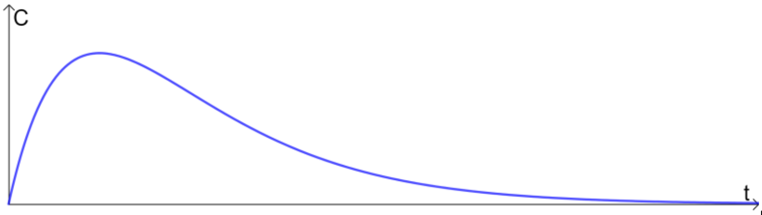

Figure 2

4pt c Compute the relative difference between the concentrations according to these two formulas at $t = 3$. Give your answer as a percentage of the concentration according to the first formula.

According to the second formula, the concentration of the drug is maximal at $t = 2$.

4pt d Use the derivative to show that this is indeed the case.

Question 4 – High distress

If you want to use the restrooms at a main railway station in the Netherlands, you have to pay 70 cents. In return, you can get a coupon, which will get you a 50 cents reduction on a purchase from one of the station shops. These coupons are valid during three months after your visit to the restrooms. Many people forget to pick up their coupon, these coupons are not considered in this question.

A statistical study has shown that 60% of the coupons that are picked up, are not used within three months.

6pt **a** Compute the probability that from a random sample of 10 picked up coupons, more than 7 of these 10 coupons are not used within three months. Give the answer rounded to four decimal places.

In recent months, Bram has regularly used the restrooms at one of the main railway stations. He therefore has collected a number of coupons which he keeps in a drawer at home. One day he takes, without worrying about the date, six of these coupons out of the drawer and puts them in his pocket. Three of these coupons are still valid, the other three are no longer valid. When paying at a station shop, he randomly picks a coupon from his pocket and, if necessary, he repeats this until he has a valid coupon.

The number of coupons that he takes from his pocket is a random variable X.

5pt **b** Show that $P(X = 2) = 0.30$ and $P(X = 3) = 0.15$.

4pt **c** Write down the complete probability distribution of X.

3pt **d** Compute $E(X)$.

Question 5 – Two growth models

A company introduces a new toy. In the first year that this toy is on sale, the number of these toys that is sold increases by 1% per week.

3pt **a** By what percentage does the number of these toys that are sold in a week increase in a year? (1 year = 52 weeks)

In the first week, 10 000 of these toys are sold. This means that the number of these toys that are sold in week *n* of the first year is given by $S(n) = 10\ 000 \cdot 1.01^{n-1}$.

4pt **b** Compute the total number of these toys that are sold in the first year according to this formula.

In the long run, the number of toys that are sold each week according to the formula given above will rise above each limit. That is why another formula is considered to predict the number of these toys that are sold per week. This formula is given by

$$S(t) = \frac{100\ 000}{2 + 8\mathrm{e}^{-0.014t}}$$

In this formula, t is the time in weeks with $t = 0$ in the week of the introduction of the new toy.

4pt **c** Compute algebraically the time at which, according to the second formula, the company will sell 25 000 of these toys per week.

3pt **d** According to the second formula, what will the number of these toys that are sold in a week be in the long run?

Question 6 – Grading exams

At the University of Sciencetown, the grading of the exams is based on the scores for the exam. First, the mean and the standard deviation of the scores of all participants are determined. Then the cut-off point between pass and fail is set at one standard deviation below the mean score. Participants whose score is more than two times the standard deviation above the mean score get the grade *Excellent*. Participants whose score is between one and two times the standard deviation above the mean will get the grade *Good*.

3000 students take part in an exam at the University of Sciencetown. Their scores are assumed to be normally distributed.

5pt **a** According to the rules of thumb, how many of these 3000 students will get the grade *Good*?

Since the pass / fail rate will always be roughly the same in this grading system, the quality of the exams has to be monitored by using other standards. One of those is the mean of the scores. In previous years, the scores were normally distributed with a mean of 550 points and a standard deviation of 90 points.
For this year, it is assumed that the scores are still normally distributed with a standard deviation of 90 points. To test whether the mean is still the same, the results of 100 randomly selected participants are examined. In this testing procedure, the significance level is $\alpha = 5\%$.

2pt **b** Formulate the null hypothesis and the alternative hypothesis for this testing procedure.

The mean score of the 100 participants in the sample is 567 points. This yields a p-value of 0.029 .

3pt **c** Find the parameters of the test statistic that is used to compute this p-value.

2pt **d** What is the conclusion of the testing procedure with this mean score in the sample? Give a motivation for your answer!

End of the exam.

Is your name on all answer sheets?

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	**function**	**derivative function**
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\frac{dk}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Logarithms

Rule	**conditions**
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \frac{a}{b}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0, \quad g \neq 1, \quad a > 0$
${}^{g}\log a = \frac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad p > 0, \quad p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \frac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* = number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If *X* and *Y* are any random variables, then: $E(X+Y)=E(X)+E(Y)$
If furthermore *X* and *Y* are independent, then: $\sigma(X+Y)=\sqrt{\sigma^2(X)+\sigma^2(Y)}$

$\sqrt{n}$-law:
For *n* independent repetitions of the same experiment where the result of each experiment is a random variable *X*, the sum of the results is a random variable *S* and the mean of the results is a random variable $\overline{X}$.

$E(S)=n\cdot E(X)$ $\quad\quad$ $\sigma(S)=\sqrt{n}\cdot\sigma(X)$

$E(\overline{X})=E(X)$ $\quad\quad$ $\sigma(\overline{X})=\dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If *X* has a binomial distribution with parameters *n* (number of experiments) and *p* (probability of success at each experiment), then

$$P(X=k)=\binom{n}{k}\cdot p^k\cdot(1-p)^{n-k} \quad \text{with} \quad k=0,1,2,\dots,n$$

Expected value: $E(X)=np$ $\quad\quad$ Standard deviation: $\sigma(X)=\sqrt{n\cdot p\cdot(1-p)}$

n and *p* are the parameters of the binomial distribution

Normal Distribution

If *X* is a normally distributed random variable with mean μ and standard deviation σ, then

$Z=\frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X<g)=P\left(Z<\frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic *T* is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are::

α	left sided	right sided	two sided
0.05	$g=\mu_T-1.645\sigma_T$	$g=\mu_T+1.645\sigma_T$	$g_l=\mu_T-1.96\sigma_T$
			$g_r=\mu_T+1.96\sigma_T$
0.01	$g=\mu_T-2.33\sigma_T$	$g=\mu_T+2.33\sigma_T$	$g_l=\mu_T-2.58\sigma_T$
			$g_r=\mu_T+2.58\sigma_T$

