CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Entrance Exam Wiskunde A

Date:	19 April 2019
Time:	13.30 – 16.30 hours
Questions:	6

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in deduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put it in your bag.

Points that can be scored for each item:						
Question	1	2	3	4	5	6
a	4	3	4	3	3	4
b	5	3	4	3	3	2
c	5	4	5	5	2	5
d	5		5	6	5	2
Total	19	10	18	17	13	13

$$\text{Grade} = \frac{\text{total points scored}}{10} + 1$$

You will pass the exam if your grade is at least 5.5 .

Question 1 – Algebraic computations

When you are asked to perform a computation ***algebraically****, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed either in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like* $\sqrt{2}$ *and* $\log(3)$.
Unless stated otherwise, all computations in this exam have to be performed algebraically.

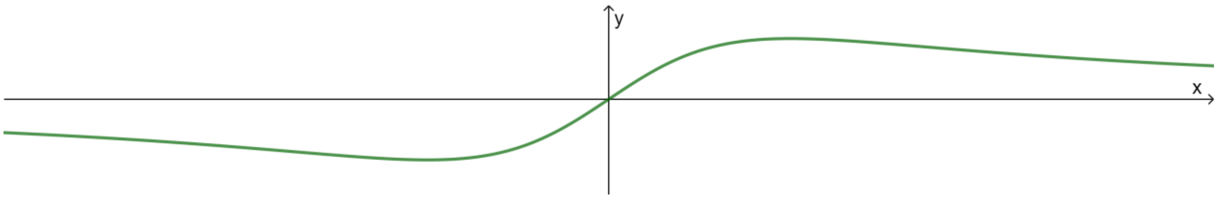
4pt a Solve the equation $2x^2 - 3x^3 = x$ algebraically.

Given are line *k* with equation $2x - 3y = 5$ and line l with equation $y = 3x - 2$.
Line m is parallel to line *k* and passes through the origin $O(0,0)$.

5pt b Compute the coordinates of the intersection of line l and line m.

In the figure below the graph is shown of the function f with function rule

$$f(x) = \frac{6x}{x^2 + 9}$$



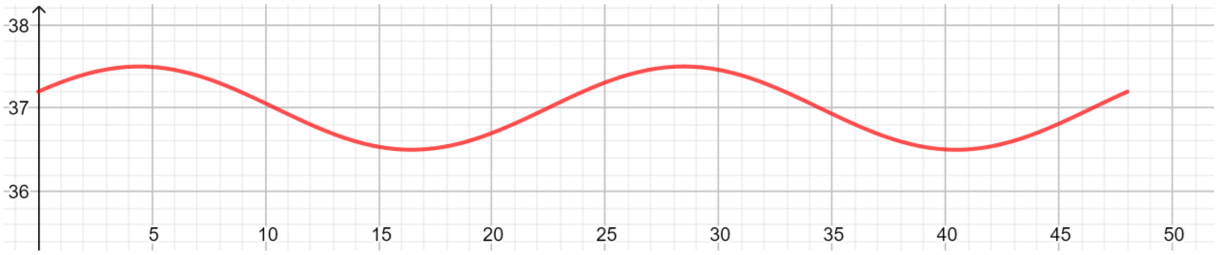
5pt c Use the derivative function of f to compute the minimal and the maximal value of $f(x)$.

5pt d Compute algebraically the coordinates of the intersection(s) of the graph of f and the graph of the function g with function rule

$$g(x) = \frac{3}{2x + 3}$$

Question 2 – A model patient

The temperature of a patient in a hospital is continuously measured during 48 hours. The result of these measurements is shown in the graph below.



The function in this figure has a formula of the form

$$T(t) = A + B\sin\big(0.262(t + 1.45)\big)$$

In this formula, T is the temperature of the patient in ℃ and t is the time in hours, with $t = 0$ at noon (12:00) of the first day.

3pt a Find the values of A and B.

3pt b Show that the number 0.262 in the formula is in accordance with the information in the graph.

4pt c Compute algebraically at what time the temperature of the patient is maximal each day. Give your answer in the format hh:mm .

Question 3 – Increasing profit

In a certain city, there is a company that is the sole vendor of commodity C.
This commodity is in high demand, so the entire supply of this commodity will be sold. However, the price at which this commodity is sold on a certain day depends on the size of the supply that is on offer on that day. This price is given by the formula

$$Price(Q) = 10 - \sqrt{3Q}$$

(the price is in euros per kg, the supply Q is in thousands of kg).
The profit (in thousands of euros) of the company is given by the formula

$$Profit(Q) = 7Q - \sqrt{3Q^3} - 6$$

At this moment, the company offers 12 000 kg of commodity C each day.

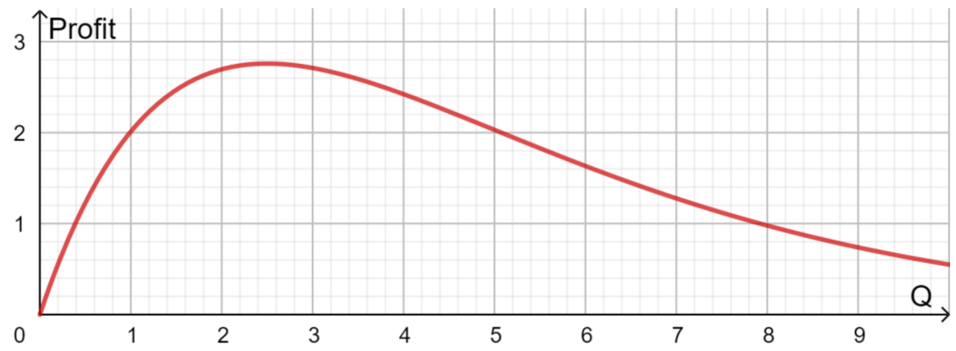
4pt **a** Compute the total revenue in euros for a day on which the supply is 12 000 kg.
Also compute the profit in euros for such a day.

4pt **b** Use the relation $Profit = Revenue - Cost$ to construct a formula for the cost as a function of Q.

The company asks you to find out whether increasing the daily supply from 12 000 kg will lead to an increase of the profit.

5pt **c** Use the derivative of the profit function to answer this question.

The profit function for another commodity is given by $Profit(Q) = 3Q \cdot \mathrm{e}^{-0.4Q}$.
The graph of this function is shown in the figure below.



In the graph, it looks as if this second profit function has a maximum at $Q = 2.5$.

5pt **d** Use the derivative of this second profit function to show that this is indeed the case.

Question 4 – Two dice

Johan has two dice: a regular die, where the outcomes 1, 2, 3, 4, 5 and 6 are all equally likely, and a four-sided die, where the outcomes 1, 2, 3 and 4 are all equally likely (see the picture on the right).

The outcome of a throw with the regular die is a random variable *X* with expected value $E(X) = 3\frac{1}{2}$ and standard deviation $\sigma(X) = \sqrt{\frac{35}{12}}$.

The outcome of a throw with the four-sided die is a random variable *Y* with expected value $E(Y) = 2\frac{1}{2}$ and standard deviation $\sigma(Y) = \sqrt{\frac{5}{4}}$.

3pt a Give a clear computation to confirm that $E(Y) = 2\frac{1}{2}$.

Johan throws both dice simultaneously. The sum of the outcomes of both dice is a random variable *S*.

3pt b Compute $\sigma(S)$.

5pt c Compute the probability that the sum of the outcomes of both dice is equal to $E(S)$.

Next, Johan throws the regular die 10 times.

6pt d Compute the probability that the outcome of the throw is not equal to 6 in at most 7 of these 10 throws.

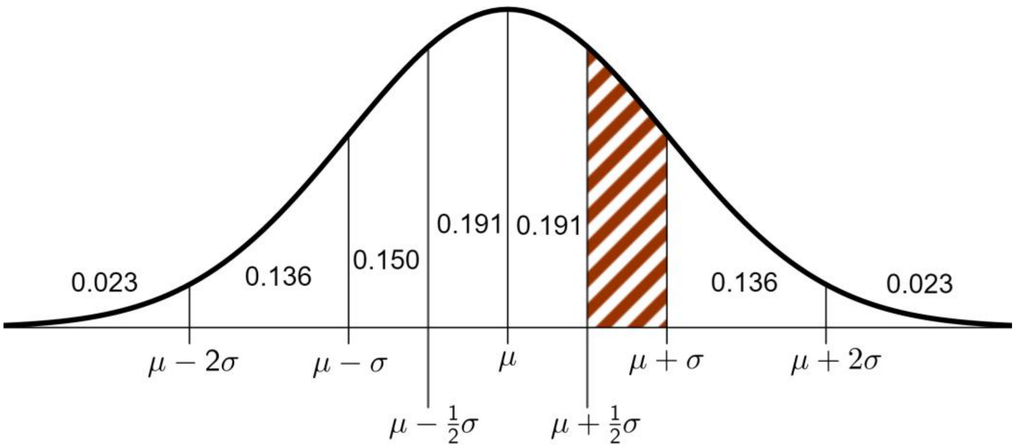
Question 5 – Under construction

For a construction job, Bert needs beams with a length of at least 200 cm.
In the village where he lives, there are two lumber stores that sell these beams.
The length of the beams that lumber store A has on offer is normally distributed with average $\mu_A = 203$ cm and standard deviation $\sigma_A = 6$ cm.
The length of the beams that lumber store B has on offer is normally distributed with average $\mu_B = 207$ cm and standard deviation $\sigma_B = 12$ cm.

Bert buys one beam from lumber store A and one beam from lumber store B.
PA is the probability that the length of the beam that Bert buys from lumber store A is more than 200 cm, PB is the probability that the length of the beam that Bert buys from lumber store B is more than 200 cm.

3pt a Use the figure below to determine PA.

3pt b Is PB larger or smaller than PA?
Explain your answer!



A normal probability distribution X
The area of the shaded region represents $P\left(\mu + \frac{1}{2}\sigma < X < \mu + \sigma\right) = 0.150$

Bert can also buy his beams from lumber store C in the next village. This store claims that the beams they have on offer, have an average length of 205 cm with a standard deviation of 10 cm. To test this claim, Bert measures the length of the beams in a random sample of 16 of these beams. He thereby assumes that the standard deviation is indeed 10 cm and he takes a significance level of $\alpha = 5\%$.

2pt c Formulate the null hypothesis and the alternative hypothesis for this testing procedure.

The mean length of the 16 beams in the sample is 200.5 cm.

5pt d What is the conclusion of the testing procedure with this mean score in the sample? Explain your answer!

Question 6 – A colony of bacteria

In this question, we study the growth of the weight of a colony of bacteria.
During the first 10 hours of this study, the weight of this colony grows exponentially. During this period, the weight is approximately given by

$$W_E(t) = 600 \cdot 1.5^t$$

In this formula, $W_E(t)$ is the weight of the colony in micrograms ($1\ \mu g = 10^{-6}\ g$) and t is the time in hours.

4pt a Compute algebraically the doubling time of the weight of this colony according to this formula. Give your answer rounded to whole minutes.

After 10 hours, the exponential growth model is no longer valid because of limitations of the available food and space. In this period, the weight of the colony is approximately given by

$$W_B(t) = 250 \cdot (700 - 1527\mathrm{e}^{-0.1t})$$

In this formula, $W_B(t)$ is the weight of the colony in micrograms and t is the time in hours.

2pt b Show that the weight of the colony at $t = 10$ according to the second formula is approximately the same as the weight of the colony at $t = 10$ according to the first formula.

5pt c Compute algebraically the time at which, according to the second formula, the weight of the colony will be equal to 0.17 g.

2pt d According to the second formula, what will the weight of the colony be in the long run?

End of the exam.

Is your name on all answer sheets?

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	**function**	**derivative function**
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \dfrac{f(x)}{g(x)}$	$q'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\dfrac{dk}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$

Logarithms

Rule	**conditions**
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0,\ g \neq 1,\ a > 0,\ b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \dfrac{a}{b}$	$g > 0,\ g \neq 1,\ a > 0,\ b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0,\ g \neq 1,\ a > 0$
${}^{g}\log a = \dfrac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0,\ g \neq 1,\ a > 0,\ p > 0,\ p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \dfrac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* = number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If *X* and *Y* are any random variables, then: $E(X+Y)=E(X)+E(Y)$
If furthermore *X* and *Y* are independent, then: $\sigma(X+Y)=\sqrt{\sigma^2(X)+\sigma^2(Y)}$

$\sqrt{n}$-law:
For *n* independent repetitions of the same experiment where the result of each experiment is a random variable *X*, the sum of the results is a random variable *S* and the mean of the results is a random variable $\overline{X}$.

$E(S)=n\cdot E(X)$ $\qquad$ $\sigma(S)=\sqrt{n}\cdot\sigma(X)$

$E(\overline{X})=E(X)$ $\qquad$ $\sigma(\overline{X})=\dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If *X* has a binomial distribution with parameters *n* (number of experiments) and *p* (probability of success at each experiment), then

$$P(X=k)=\binom{n}{k}\cdot p^k\cdot(1-p)^{n-k} \quad \text{with} \quad k=0,1,2,\ldots,n$$

Expected value: $E(X)=np$ $\qquad$ Standard deviation: $\sigma(X)=\sqrt{n\cdot p\cdot(1-p)}$

n and *p* are the parameters of the binomial distribution

Normal Distribution

If *X* is a normally distributed random variable with mean μ and standard deviation σ, then

$Z=\frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X<g)=P\left(Z<\frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic *T* is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are::

α	left sided	right sided	two sided
0.05	$g=\mu_T-1.645\sigma_T$	$g=\mu_T+1.645\sigma_T$	$g_l=\mu_T-1.96\sigma_T$
			$g_r=\mu_T+1.96\sigma_T$
0.01	$g=\mu_T-2.33\sigma_T$	$g=\mu_T+2.33\sigma_T$	$g_l=\mu_T-2.58\sigma_T$
			$g_r=\mu_T+2.58\sigma_T$

