CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Entrance Exam Wiskunde A

Date: 20 July 2020
Time: 3 hours
Questions: 7

Please read the instructions below carefully before answering the questions. Failing to comply with these instructions may result in deduction of points.

Make sure your name is clearly written on every answer sheet.

Take a new answer sheet for every question.

Show all your calculations clearly. Illegible answers and answers without a calculation or an explanation of the use of your calculator are invalid (*see also question 1)*.

Write your answers in ink. Do not use a pencil, except when drawing graphs. Do not use correction fluid.

You can use a basic scientific calculator. **Other equipment, like a graphing calculator, a calculator with the option of computing integrals, a formula chart, BINAS or a book with tables, is NOT permitted.**

On the last two pages of this exam you will find a list of formulas.

You can use a dictionary if it is approved by the invigilator.

Please **switch off your mobile telephone** and put it in your bag.

Points that can be scored for each item:

Question	1	2	3	4	5	6	7
a	4	6	3	4	4	5	3
b	6	5	3	5	2	1	4
c	5	4	4		4	6	5
d	5						2
Total	20	15	10	9	10	12	14

$$\text{Grade} = \frac{\text{total points scored}}{10} + 1$$

You will pass the exam if your grade is at least 5.5 .

Question 1 – Algebraic computations

*When you are asked to perform a computation **algebraically**, your computation should be fully worked out on paper. Reading function values from a table (including tables produced by a calculator) is not allowed either in algebraic calculations. You can use a calculator for simple calculations and for approximations of numbers like* $\sqrt{2}$ *and* $\log(3)$.
Unless stated otherwise, all computations in this exam have to be performed algebraically.

The function f is given by

$$f(x) = 5 - 2x - 4x^2$$

4pt a Compute algebraically the coordinates of the common point(s) of the graph of f and the straight line with equation $y = 10x + 14$.

l is the tangent line to the graph of f in point $A(0{,}5)$.
m is the line with equation $x - 4y = 4$.

6pt b Compute algebraically the *x*-coordinate of the intersection of lines l and m.

The function g is given by

$$g(x) = \frac{x + 2}{2x + 10}$$

There are two points on the graph of g at which the slope of the tangent line to this graph equals $\frac{1}{6}$.

5pt c Use the derivative function of g to compute the *x*-coordinates of these two points algebraically.

The function h is given by

$$h(x) = 3 + 4 \cdot \log(5x)$$

5pt d Compute algebraically the value of x for which $h(x) = 11$.

Question 2 – Playing with maximal profit

When a new toy is brought to market, the profit on the sale of this toy first increases because many children want to try this new toy, but later the profit decreases when the novelty of the toy has faded.

The Gelo toy company wants to bring a new toy to market and their marketing department has developed two models for the profit on the sale of this toy as a function of time.

In the first model, the profit is given by the formula

$$P_1(t) = 450t - 25t \cdot \sqrt{t}$$

In this formula, t is the time in days with $t = 0$ at the moment the toy is brought to market, and $P_1(t)$ is the daily profit in euros.

6pt a Compute algebraically the maximal daily profit of the Gelo toy company according to this first model.

In the second model, the profit is given by the formula

$$P_2(t) = 7.7t^2 \cdot \mathrm{e}^{-t/72}$$

In this formula, t is again the time in days with $t = 0$ at the moment the toy is brought to market, and $P_2(t)$ is the daily profit in euros.

5pt b Use the derivative function of $P_2(t)$ to show that according to this second model the daily profit of the Gelo toy company is maximal at $t = 144$.

From experience, the Gelo company knows that two years after the introduction, there is still some profit left. However, four years after the new toy is brought to market, the profit is practically zero. *Use 1 year = 365 days.*

4pt c Does one of these models match this experience?
Use a computation to explain your answer.

Question 3 – Bicycle wheels

In a bicycle factory the quality of bicycle wheels is tested.

In a first test, a sensor is placed on the spoke of a wheel and the wheel is rotated at a constant speed around its axis. The distance in centimeters between the sensor and the floor as a function of time in seconds, is given by the formula

$$H(t) = 135 + 30\sin(10\pi t)$$

3pt a Compute algebraically the minimal and the maximal value of $H(t)$.

3pt b Compute algebraically how many times per second the wheel turns around its axis.

In another test, the sensor is placed on a spoke of a second wheel. This wheel is also rotated at a constant speed around its axis. At the start of the test ($t = 0$), the sensor is at 123 cm above the floor. At $t = 0.07$, the sensor is for the first time at its highest point, which is 140 cm above the floor. And at $t = 0.21$, the sensor is for the first time at its lowest point, which is 106 cm above the floor.

The distance in centimeters between the sensor and the floor as a function of time in seconds is given by a formula of the form $H(t) = A + B\sin(Ct)$.

4pt c Find values of A, B and C that match the description given above.

Question 4 – A day at the beach

Annie (female, age 18), Bert (male, age 19), Carla (female, age 20), Duncan (male, age 19), Erica (female, age 18), Freddie (male, age 18), Gina (female, age 20) and Ivan (male, age 19) are a group of 8 friends. On a nice day, they want to go to the beach together. However, they have a car available with room for only 5 of them. That means that 3 of these friends have to make their way to the beach by public transport. They decide draw lots to determine who has to go by public transport.

4pt a Compute the probability that exactly 2 of the male friends have to go by public transport.

5pt b Compute the probability that the average age of the 3 friends who go by public transport is exactly 19.

Question 5 – WADA

The World Anti-Doping Agency WADA monitors about 250 000 in- and out-of-competition doping tests per year. These tests are performed by various accredited doping labs across the world.

In this question, we focus on a fictional doping lab which performs a specific out-of-competition test 10 000 times per year. They want to minimize the number of false accusations and therefore opt for a test with a high specificity. If an athlete is clean, the test gives a negative result in 96% of the cases. A high specificity often comes at the expense of the sensitivity of a test. For this test, if an athlete has used doping, the test gives a positive result in 64% of the cases.
Furthermore, we assume that in 2% of the tests the tested athlete has used doping.

4pt **a** Copy the contingency table below on your answer sheet and fill out the missing entries.

	Positive test	Negative test	
Used doping			200
Clean			
			10 000

2pt **b** Compute the probability that an athlete who tests positive indeed has used doping.

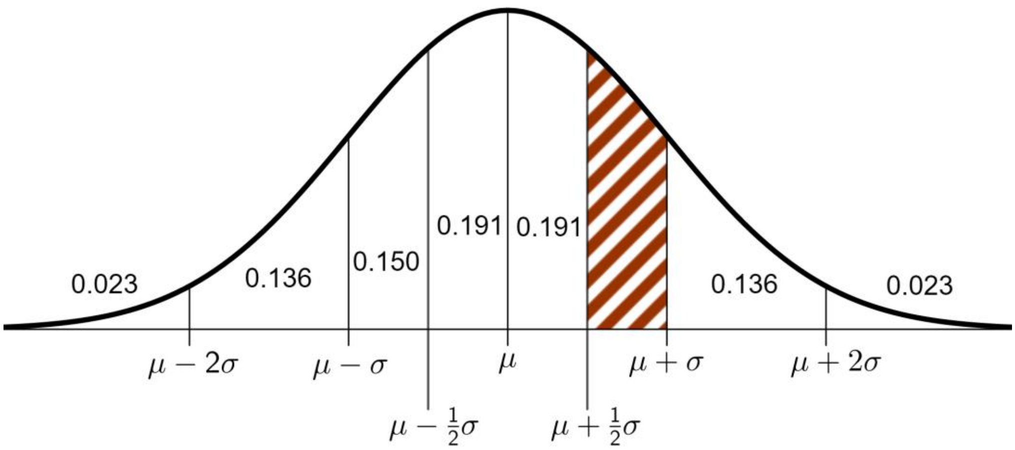
WADA regularly checks the performance of the accredited doping labs. At one of these checks, WADA randomly selects 100 tests from various labs for further inspection. We still assume that in 2% of the tests the tested athlete has used doping.

4pt **c** Compute the probability that the tested athlete has used doping in exactly 3 of these 100 tests.

Question 6 – Weighing cheeses

Bert runs a dairy farm in the beautiful region of Waterland, just North of Amsterdam. There, he produces Edam cheeses. Last year, he produced 2500 of these cheeses. The weight of the cheeses was normally distributed with an average of 1110 g and a standard deviation of 20 g. Most of these cheeses were bought by a local supermarket chain. For uniformity reasons, this chain only bought the cheeses with a weight between 1100 g and 1150 g.

5pt **a** Use the figure below to compute the number of Bert's cheeses that were bought by the supermarket chain last year.



A normal probability distribution X
The area of the shaded region represents $P\left(\mu + \frac{1}{2}\sigma < X < \mu + \sigma\right) = 0.150$

This year, Bert wants the weight of the cheeses to better match the wishes of the supermarket chain. Therefore, he changes his recipe with the aim to increase the average weight of his cheeses. To test whether this change in recipe has the desired effect, he measures the weight of 25 cheeses that are produced with this new recipe. For this testing procedure, he takes a significance level of $\alpha = 0.05$ and he assumes that the standard deviation of the weight of the cheeses still is 20 g.

1pt **b** What is the null hypothesis and what is the alternative hypothesis of this testing procedure?

6pt **c** What is the result of this testing procedure if the average weight of these 25 cheeses is 1120 g?

Question 7 – E-bikes

On Windhill Island, bicycles were never popular because of the windy conditions and the hilly landscape. Until, in 2010, e-bikes were introduced. From July 2010 until July 2020, the number of e-bikes has increased by 40% per year. At this moment, there are 70 000 e-bikes on the island.

3pt a Compute algebraically how many e-bikes there were on this island in July 2010.

4pt b Compute algebraically the month in which there would be 90 000 e-bikes on Windhill Island if the number of e-bikes continued to increase by 40% per year.

Since a large part of the inhabitants now has an e-bike, predictions are that the number of e-bikes will not increase by 40% per year from now on. Instead, the predicted growth of the number of e-bikes is modelled by the formula

$$N(t) = 5 \cdot (24 - 10 \cdot \mathrm{e}^{-0.04t})$$

In this formula, $N(t)$ is the number of e-bikes in thousands and t is the time in months, with $t = 0$ on 1 July 2020.

5pt c Compute algebraically the month in which according to this formula there will be 90 000 e-bikes on Windhill Island.

2pt d According to this formula, what could be the maximal number of e-bikes on Windhill Island?

End of the exam.

Is your name on all answer sheets?

Formula list wiskunde A

Quadratic equations

The solutions of the equation $ax^2 + bx + c = 0$ with $a \neq 0$ and $b^2 - 4ac \geq 0$ are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Differentiation

Rule	**function**	**derivative function**
Sum rule	$s(x) = f(x) + g(x)$	$s'(x) = f'(x) + g'(x)$
Product rule	$p(x) = f(x) \cdot g(x)$	$p'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule	$q(x) = \frac{f(x)}{g(x)}$	$q'(x) = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\left(g(x)\right)^2}$
Chain rule	$k(x) = f(g(x))$	$k'(x) = f'\left(g(x)\right) \cdot g'(x)$ or $\frac{dk}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

Logarithms

Rule	**conditions**
${}^{g}\log a + {}^{g}\log b = {}^{g}\log ab$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a - {}^{g}\log b = {}^{g}\log \frac{a}{b}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad b > 0$
${}^{g}\log a^p = p \cdot {}^{g}\log a$	$g > 0, \quad g \neq 1, \quad a > 0$
${}^{g}\log a = \frac{{}^{p}\log a}{{}^{p}\log g}$	$g > 0, \quad g \neq 1, \quad a > 0, \quad p > 0, \quad p \neq 1$

Arithmetic and geometric sequences

Arithmetic sequence:	$Sum = \frac{1}{2} \cdot number\ of\ terms \cdot (u_e + u_l)$
Geometric sequence:	$Sum = \frac{u_{l+1} - u_e}{r - 1} \quad (r \neq 1)$
In both formulas:	*e* = number first term of the sum; *l* = number last term of the sum

More formulas on the next page.

Formula list wiskunde A (continued)

Probability

If *X* and *Y* are any random variables, then: $E(X+Y) = E(X) + E(Y)$
If furthermore *X* and *Y* are independent, then: $\sigma(X+Y) = \sqrt{\sigma^2(X) + \sigma^2(Y)}$

$\sqrt{n}$-law:
For *n* independent repetitions of the same experiment where the result of each experiment is a random variable *X*, the sum of the results is a random variable *S* and the mean of the results is a random variable $\overline{X}$.

$E(S) = n \cdot E(X)$ $\qquad$ $\sigma(S) = \sqrt{n} \cdot \sigma(X)$

$E(\overline{X}) = E(X)$ $\qquad$ $\sigma(\overline{X}) = \dfrac{\sigma(X)}{\sqrt{n}}$

Binomial Distribution

If *X* has a binomial distribution with parameters *n* (number of experiments) and *p* (probability of success at each experiment), then

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \text{with} \quad k = 0, 1, 2, \ldots, n$$

Expected value: $E(X) = np$ $\qquad$ Standard deviation: $\sigma(X) = \sqrt{n \cdot p \cdot (1-p)}$

n and *p* are the parameters of the binomial distribution

Normal Distribution

If *X* is a normally distributed random variable with mean μ and standard deviation σ, then

$Z = \frac{X-\mu}{\sigma}$ has a standard normal distribution and $P(X < g) = P\left(Z < \frac{g-\mu}{\sigma}\right)$

μ and σ are the parameters of the normal distribution.

Hypothesis testing

In a testing procedure where the test statistic *T* is normally distributed with mean μ_T standard deviation σ_T the boundaries of the rejection region (the critical region) are:

α	left sided	right sided	two sided
0.05	$g = \mu_T - 1.645\sigma_T$	$g = \mu_T + 1.645\sigma_T$	$g_l = \mu_T - 1.96\sigma_T$
			$g_r = \mu_T + 1.96\sigma_T$
0.01	$g = \mu_T - 2.33\sigma_T$	$g = \mu_T + 2.33\sigma_T$	$g_l = \mu_T - 2.58\sigma_T$
			$g_r = \mu_T + 2.58\sigma_T$

