# CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

## Answers & brief elaborations Wiskunde B 19 December 2018

1a There is a vertical tangent line if $x'(t) = 0$ and $y'(t) \neq 0$.

$x'(t) = 0 \Leftrightarrow 8t^3 - 2t = 0 \Leftrightarrow 2t(4t^2 - 1) = 0 \Leftrightarrow t = 0 \vee t = \frac{1}{2} \vee t = -\frac{1}{2}$

$y'(t) = 0 \Leftrightarrow 3t^2 - 3 = 0 \Leftrightarrow t^2 = 1 \Leftrightarrow t = \pm 1$

$t = 0 \;\Rightarrow\; x = 0;\; y = 0$

$t = \frac{1}{2} \;\Rightarrow\; x = \frac{2}{16} - \frac{1}{4} = -\frac{1}{8};\;\; y = \frac{1}{8} - \frac{3}{2} = -\frac{11}{8}$

$t = -\frac{1}{2} \;\Rightarrow\; x = \frac{2}{16} - \frac{1}{4} = -\frac{1}{8};\;\; y = -\frac{1}{8} + \frac{3}{2} = \frac{11}{8}$

1b $x'(t) = 8t^3 - 2t \Rightarrow x'(2) = 8 \cdot 8 - 2 \cdot 2 = 60$

$y'(t) = 3t^2 - 3 \;\Rightarrow y'(2) = 3 \cdot 4 - 3 = 9$

$v = \sqrt{\left(x'(2)\right)^2 + \left(y'(2)\right)^2} = \sqrt{60^2 + 9^2} = \sqrt{3681}$

1c $y(t) = 0 \Leftrightarrow t^3 - 3t = 0 \Leftrightarrow t = 0 \vee t^2 = 3 \Leftrightarrow t = 0 \vee t = \sqrt{3} \vee t = -\sqrt{3}$

In the intersection with the positive *x*-axis we have $t = \sqrt{3} \vee t = -\sqrt{3}$

$t = \sqrt{3}$ yields $\overrightarrow{v_1} = \begin{pmatrix} x'(\sqrt{3}) \\ y'(\sqrt{3}) \end{pmatrix} = \begin{pmatrix} 22\sqrt{3} \\ 6 \end{pmatrix}$

$t = -\sqrt{3}$ yields $\overrightarrow{v_2} = \begin{pmatrix} x'(-\sqrt{3}) \\ y'(-\sqrt{3}) \end{pmatrix} = \begin{pmatrix} -22\sqrt{3} \\ 6 \end{pmatrix}$

$\cos(\alpha) = \dfrac{(\overrightarrow{v_1}, \overrightarrow{v_2})}{|v_1| \cdot \lceil v_2 \rceil} = \dfrac{-484 \cdot 3 + 36}{\left(\sqrt{484 \cdot 3 + 36}\right)^2} = -0{,}952 \Rightarrow \alpha = 162°$

The angle between the two branches of the path is therefore $180° - 162° = 18°$

*1c* *Alternative*

$t = \sqrt{3} \;\;\Rightarrow\;\; \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y'(\sqrt{3})}{x'(\sqrt{3})} = \dfrac{3 \cdot 3 - 3}{8 \cdot 3\sqrt{3} - 2\sqrt{3}} = \dfrac{6}{22\sqrt{3}}$

The angle between the increasing branch with the positive *x*-axis is $\tan^{-1}\left(\frac{6}{22\sqrt{3}}\right) = 8{,}9°$

The angle between the two branches of the path is therefore $2 \times 8{,}9° \approx 18°$


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2a The centre of $c_2$ is the midpoint of *BC*, that is $M(8,5)$

The radius of $c_2$ is $|MB| = \frac{1}{2}|BC| = 6$

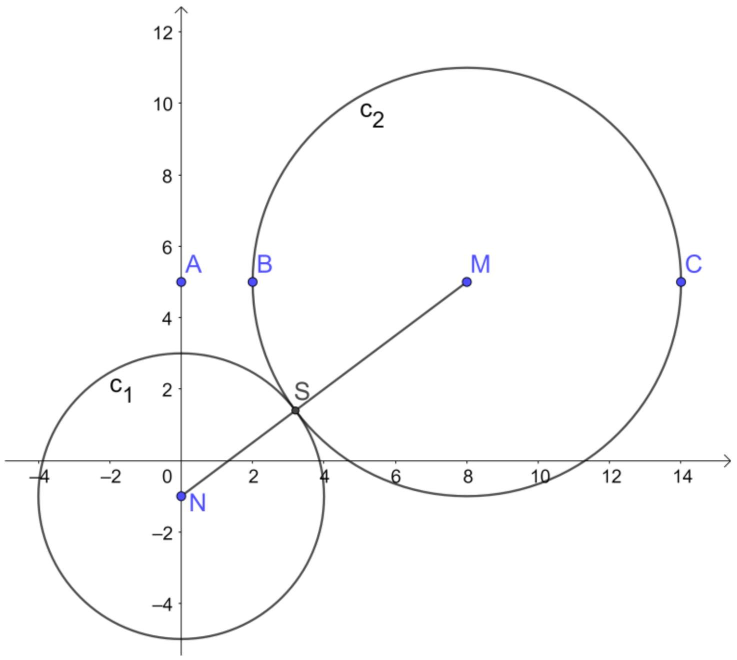
$(x-a)^2 + (y-b)^2 = r^2$ with $a = 8$, $b = 5$, $r = 6$ yields $(x-8)^2 + (y-5)^2 = 36$

*2a* *Alternative*

The radius of $c_2$ is $r = \frac{1}{2}|BC| = 6$

Substitution of the coordinates of *B* and *C* in the equation $(x-a)^2 + (y-b)^2 = 36$ yields a system of two equations with solution $a = 8$ and $b = 5$.

2b




The equation of $c_1$ can be written as $x^2 + (y+1)^2 = 16$

$c_1$ is therefore the circle with centre $N(0,-1)$ and radius $r_1 = 4$

$c_2$ is the circle with centre $M(8,5)$ and radius $r_2 = 6$

$|MN| = \sqrt{8^2 + 6^2} = 10 = 4 + 6 = r_1 + r_2$

Therefore, these circles intersect in point *S*

*2b* *Alternative 1*

Find a vector representation or an equation of the line through *M* and *N*.

Compute the intersections with $c_1$.

Show that the intersection $S(3.2\,,1.4)$ is also on $c_2$ .

*2b* *Alternative 2*

$$\begin{cases} x^2 + y^2 + 2y - 15 = 0 \\ (x-8)^2 + (y-5)^2 = 36 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 + 2y - 15 = 0 \\ x^2 - 16x + 64 + y^2 - 10y + 25 = 36 \end{cases} \Leftrightarrow$$

$$\begin{cases} x^2 + y^2 + 2y - 15 = 0 \\ x^2 - 16x + y^2 - 10y + 53 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 + 2y - 15 = 0 \\ 16x + 12y - 68 = 0 \end{cases} \Leftrightarrow \begin{cases} x^2 + y^2 + 2y - 15 = 0 \\ x = -\frac{3}{4}y + \frac{17}{4} \end{cases}$$

Substitution of the second equation in the first yields

$\frac{9}{16}y^2 - \frac{102}{16}y + \frac{289}{16} + y^2 + 2y - 15 = 0 \Leftrightarrow \frac{25}{16}y^2 - \frac{70}{16}y + \frac{49}{16} = 0 \Leftrightarrow y = \frac{7}{5}$,

so $S(3.2\,,1.4)$ is the only common point of $c_1$ and $c_2$ .

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2c The equation of $c_1$ can be written as $x^2 + (y+1)^2 = 16$

$c_1$ is therefore the circle with centre $N(0,-1)$ and radius 4.

The distance between the tangent lines through point *A* and centre $N(0,-1)$ is therefore 4. These lines have an equation of the form $y = ax + 5 \Leftrightarrow ax - y = -5$

The distance between a point $(x_0, y_0)$ and a line with equation $ax + by = c$ is given by $\frac{|ax_0 + by_0 - c|}{\sqrt{a^2+b^2}}$ . This must be equal to 4.

Substitution of $b = -1$, $c = -5$, $x_0 = 0$ and $y_0 = -1$ then yields

$$\frac{|0 + (-1)\cdot(-1) + 5|}{\sqrt{a^2+1}} = 4 \Leftrightarrow \sqrt{a^2+1} = \tfrac{6}{4} = \tfrac{3}{2} \Leftrightarrow a^2 + 1 = \tfrac{9}{4} \Leftrightarrow a^2 = \tfrac{5}{4} \Leftrightarrow a = \pm\tfrac{1}{2}\sqrt{5}$$

The tangent lines are $y = \frac{1}{2}\sqrt{5}\cdot x + 5$ and $y = -\frac{1}{2}\sqrt{5}\cdot x + 5$

*2c Alternative 1*

A line through $A(0,5)$ has equation $y = ax + 5$. This is a tangent line to $c_1$ if the system $\begin{cases} x^2 + y^2 + 2y - 15 = 0 \\ y = ax + 5 \end{cases}$ has exactly one solution.

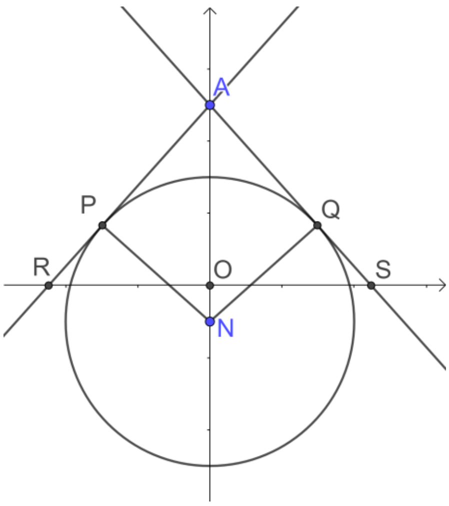
$x^2 + (ax+5)^2 + 2(ax+5) - 15 = 0 \Leftrightarrow (1+a^2)x^2 + 12ax + 20 = 0$

$D = (12a)^2 - 4\cdot(1+a^2)\cdot 20 = 64a^2 - 80$

$D = 0 \Leftrightarrow a^2 = \frac{80}{64} = \frac{5}{4} \Leftrightarrow a = \pm\frac{1}{2}\sqrt{5}$

The tangent lines are $y = \frac{1}{2}\sqrt{5}\cdot x + 5$ and $y = -\frac{1}{2}\sqrt{5}\cdot x + 5$

*2c Alternative 2*




$\angle NPA = 90°$', $|AN| = 6$, $|PN| = 4 \Rightarrow |AP| = \sqrt{6^2 - 4^2} = \sqrt{20}$

$\tan(\angle PAN) = \frac{4}{\sqrt{20}}$; $\tan(\angle ORA) = \frac{1}{\tan(\angle PAN)} = \frac{\sqrt{20}}{4} = \frac{1}{2}\sqrt{5}$

The equation of the line through *A* and *P* is therefore $y = \frac{1}{2}\sqrt{5}\cdot x + 5$

The equation of the line through *A* and *Q* is $y = -\frac{1}{2}\sqrt{5}\cdot x + 5$

3a $f$ has a vertical asymptote if the denominator is 0 and the numerator is not.

$(2x+1)(x^2-4)=0 \Leftrightarrow 2x=-1 \vee x^2=4 \Leftrightarrow x=-\frac{1}{2} \vee x=\pm 2$

$3x^2-6x=0 \Leftrightarrow 3x(x-2)=0 \Leftrightarrow x=0 \vee x=2$

Vertical asymptotes: $x=-\frac{1}{2},\ x=-2$

$$f(x)=\frac{3x^2-6x}{2x^3+x^2-8x-4}=\frac{\frac{3}{x}-\frac{6}{x^2}}{2+\frac{1}{x}-\frac{8}{x^2}-\frac{4}{x^3}}\to\frac{0-0}{2+0-0-0}=0\ \ (x\to\pm\infty)$$

Horizontal asymptote: $x=0$

3b For $x\neq 2$ we have:

$$f(x)=\frac{3x^2-6x}{(2x+1)(x^2-4)}=\frac{3x(x-2)}{(2x+1)(x+2)(x-2)}=\frac{3x}{(2x+1)(x+2)}$$

$$g(x)=\frac{2(2x+1)}{(x+2)(2x+1)}-\frac{x+2}{(2x+1)(x+2)}=\frac{2(2x+1)-(x+2)}{(x+2)(2x+1)}=\frac{4x+2-x-2}{(x+2)(2x+1)}$$

3c $$g'(x)=-\frac{2}{(x+2)^2}+\frac{2}{(2x+1)^2}$$

$g'(x)=0 \Leftrightarrow (x+2)^2=(2x+1)^2 \Leftrightarrow x^2+4x+4=4x^2+4x+1$

$\Leftrightarrow -3x^2+3=0 \Leftrightarrow x=\pm 1$

Maximum on interval $0\leq x\leq 3$: $g(1)=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$

$h'(x)=\frac{1}{3}\mathrm{e}^{1-x}-\frac{1}{3}x\mathrm{e}^{1-x}$; $h'(x)=0 \Leftrightarrow (1-x)\mathrm{e}^{1-x}=0 \Leftrightarrow x=1$

Maximum on interval $0\leq x\leq 3$: $h(1)=\frac{1}{3}\cdot 1\cdot \mathrm{e}^0=\frac{1}{3}$

3d $$\int_0^2 g(x)\ \mathrm{d}x=\left[2\ln(x+2)-\tfrac{1}{2}\ln(2x+1)\right]_0^2=2\ln(4)-\tfrac{1}{2}\ln(5)-2\ln(2)+\tfrac{1}{2}\ln(1)$$

$$=2\ln(4)-\ln(2^2)-\tfrac{1}{2}\ln(5)+\tfrac{1}{2}\cdot 0=\ln(4)-\ln(\sqrt{5})=\ln\left(\frac{4}{\sqrt{5}}\right)$$

Other simplifications possible, such as $\frac{1}{2}\ln\left(\frac{16}{5}\right)=\ln(\sqrt{3.2})$

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4a $g_p'(x) = -2p\mathrm{e}^{px} \Rightarrow g_p'(0) = -2p;\ \ h'(x) = 2\mathrm{e}^x \Rightarrow h'(0) = 2$

$g_p'(0) \cdot h'(0) = -1 \Leftrightarrow -2p \cdot 2 = -1 \Leftrightarrow p = \frac{1}{4}$

4b $g_p(\ln(2)) = 4 - 2\mathrm{e}^{p\ln(2)} = 4 - 2\left(\mathrm{e}^{\ln(2)}\right)^p = 4 - 2 \cdot 2^p$

$g_1(\ln(2)) = 4 - 2 \cdot \mathrm{e}^{\ln(2)} = 4 - 2 \cdot 2 = 0$

$g_p(\ln(2)) - g_1(\ln(2)) = 8 \Leftrightarrow 4 - 2 \cdot 2^p - 0 = 8 \Leftrightarrow 2^p = -2$

This has no solution.

$g_1(\ln(2)) - g_p(\ln(2)) = 8 \Leftrightarrow 0 - 4 + 2 \cdot 2^p = 8 \Leftrightarrow 2^p = 6 \Leftrightarrow p = {}^2\log(6)$

4c For $x > 0$ we have $g_1(x) > g_2(x)$, therefore we have to compute:

$$\int_0^q g_1(x) - g_2(x)\,\mathrm{d}x = \int_0^q 4 - 2\mathrm{e}^x - (4 - 2\mathrm{e}^{2x})\ \mathrm{d}x = \int_0^q 2\mathrm{e}^{2x} - 2\mathrm{e}^x\ \mathrm{d}x$$

$$= \left[\mathrm{e}^{2x} - 2\mathrm{e}^x\right]_0^q = \mathrm{e}^{2q} - 2\mathrm{e}^q - 1 + 2$$

4d $\mathrm{e}^{2q} - 2\mathrm{e}^q + 1 = 4 \Leftrightarrow (\mathrm{e}^q)^2 - 2\mathrm{e}^q - 3 = 0 \Leftrightarrow (\mathrm{e}^q + 1)(\mathrm{e}^q - 3) = 0 \Leftrightarrow \mathrm{e}^q = -1 \vee \mathrm{e}^q = 3$

$\mathrm{e}^q = -1$ has no solution; $\mathrm{e}^q = 3 \Leftrightarrow q = \ln(3)$

5a $f(x) = -1 \Leftrightarrow \sin\left(2x - \frac{1}{3}\pi\right) = -\frac{1}{2} \Leftrightarrow \sin\left(2x - \frac{1}{3}\pi\right) = \sin\left(-\frac{1}{6}\pi\right)$

This yields $2x - \frac{1}{3}\pi = -\frac{1}{6}\pi + k \cdot 2\pi \Leftrightarrow 2x = \frac{1}{6}\pi + k \cdot 2\pi \Leftrightarrow x = \frac{1}{12}\pi + k \cdot \pi$

or $2x - \frac{1}{3}\pi = 1\frac{1}{6}\pi + k \cdot 2\pi \Leftrightarrow 2x = 1\frac{1}{2}\pi + k \cdot 2\pi \Leftrightarrow x = \frac{3}{4}\pi + k \cdot \pi$

Solutions with $0 \le x \le 2\pi$: $\frac{1}{12}\pi,\ \frac{3}{4}\pi,\ 1\frac{1}{12}\pi$ en $1\frac{3}{4}\pi$

5b $f'(x) = 4\cos\left(2x - \frac{1}{3}\pi\right) \Rightarrow f'\left(\frac{1}{6}\pi\right) = 4\cos(0) = 4$

Tangent line $y = 4\left(x - \frac{1}{6}\pi\right) \Leftrightarrow y = 4x - \frac{2}{3}\pi$, so *B* is the point $\left(0, -\frac{2}{3}\pi\right)$

Area $\Delta OAB = \frac{1}{2} \cdot |OA| \cdot |OB| = \frac{1}{2} \cdot \frac{1}{6}\pi \cdot \frac{2}{3}\pi = \frac{1}{18}\pi^2$

5c $L(q) = g(q) - f(q) = 4 - 2\sin(2q) - 2\sin\left(2q - \frac{1}{3}\pi\right)$

$L'(q) = -4\cos(2q) - 4\cos\left(2q - \frac{1}{3}\pi\right)$

$L'(q) = 0 \Leftrightarrow \cos(2q) = -\cos\left(2q - \frac{1}{3}\pi\right) \Leftrightarrow \cos(2q) = \cos\left(2q - \frac{1}{3}\pi + \pi\right)$

This has one possible continuation:

$2q = -\left(2q + \frac{2}{3}\pi\right) + k \cdot 2\pi \Leftrightarrow 4q = -\frac{2}{3}\pi + k \cdot 2\pi \Leftrightarrow q = -\frac{1}{6}\pi + k \cdot \frac{1}{2}\pi$

$L\left(-\frac{1}{6}\pi\right) = 4 - 2\sin\left(-\frac{1}{3}\pi\right) - 2\sin\left(-\frac{2}{3}\pi\right) = 4 + \sqrt{3} + \sqrt{3} = 4 + 2\sqrt{3};$

$L\left(\frac{1}{3}\pi\right) = 4 - 2\sin\left(\frac{2}{3}\pi\right) - 2\sin\left(\frac{1}{3}\pi\right) = 4 - \sqrt{3} - \sqrt{3} = 4 - 2\sqrt{3}$

The minimal distance is therefore $4 - 2\sqrt{3}$


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