CENTRALE COMMISSIE VOORTENTAMEN WISKUNDE

Worked Solutions Wiskunde B 19 April 2019

1a $y(t) = x(t) + 7 \Leftrightarrow -t^2 + 4 = t^2 - 2t - 3 + 7$
Hence $2t^2 - 2t = 0 \Leftrightarrow 2t(t-1) = 0 \Leftrightarrow t = 0 \vee t = 1$
$t = 0$ yields $x = -3$ en $y = 4$, so the first intersection is $(-3,4)$.
$t = 1$ yields $x = -4$ en $y = 3$, so the second intersection is $(-4,3)$
The distance is $\sqrt{(-4 - -3)^2 + (3-4)^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

1b $y(t) = 0 \Leftrightarrow t^2 = 4 \Leftrightarrow t = 2 \vee t = -2$
$t = -2$ yields $x = 5$; $t = 2$ yields $x = -3$, hence *A* is the point $(5,0)$.
$x'(t) = 2t - 2$, so $x'(-2) = -6$; $y'(t) = -2t$, so $y'(-2) = 4$
The vector representation of the tangent line is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 4 \end{pmatrix}$

1c $v(t) = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} = \sqrt{(2t-2)^2 + (-2t)^2}$
$= \sqrt{4t^2 - 8t + 4 + 4t^2} = \sqrt{8t^2 - 8t + 4}$
$v'(t) = \dfrac{16t - 8}{2\sqrt{8t^2 - 8t + 4}}$
$v'(t) = 0 \Leftrightarrow t = \frac{1}{2}$
$v\left(\frac{1}{2}\right) = \sqrt{8 \cdot \frac{1}{4} - 8 \cdot \frac{1}{2} + 4} = \sqrt{2}$

2a With $C(x,0)$ we have $|AC| = \sqrt{(x-1)^2 + (0-2)^2} = \sqrt{x^2 - 2x + 5}$
and $|BC| = \sqrt{(x-3)^2 + (0-8)^2} = \sqrt{x^2 - 6x + 9 + 64} = \sqrt{x^2 - 6x + 73}$
$|AC| = |BC| \Leftrightarrow -2x + 5 = -6x + 73 \Leftrightarrow 4x = 68 \Leftrightarrow x = 17$

2a *Alternative 1*
C is the intersection of the line segment bisector of *AB* and the de *x*-axis.
This line segment bisector passes through point $\left(\frac{3+1}{2}, \frac{8+2}{2}\right) = (2,5)$.
The slope of *AB* is $\frac{8-2}{3-1} = 3$, so the slope of the line segment bisector is $-\frac{1}{3}$.
Hence, the equation of the line segment bisector is $y - 5 = -\frac{1}{3}(x-2)$.
$y = 0$ then yields $-5 = -\frac{1}{3}(x-2) \Leftrightarrow x - 2 = 15 \Leftrightarrow x = 17$

2a *Alternative 2*
The direction vector *AB* is $\begin{pmatrix} 3-1 \\ 8-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$. The direction vector of the line segment bisector of *AB* is a normal vector thereof, e.g. $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$. The midpoint of *AB* is $(2,5)$.
So a vector representation of the line segment bisector of *AB* is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -2 \end{pmatrix}$
$y = 0$ then yields $0 = 5 - 2\lambda \Leftrightarrow \lambda = \frac{5}{2}$ and $x = 2 + 6\lambda = 2 + 15 = 17$

2b The line $y = ax$ touches the circle in $O(0{,}0)$ if this is the only common point of the line and the circle.
Substitution of $y = ax$ into $x^2 + y^2 - 2x - 4y = 0$ yields
$x^2 + a^2x^2 - 2x - 4ax = 0 \Leftrightarrow (1 + a^2)x^2 + (-2 - 4a)x = 0$
$\Leftrightarrow x\big((1 + a^2)x - 2 - 4a\big) = 0 \Leftrightarrow x = 0 \vee x = \dfrac{2 + 4a}{1 + a^2}$
There is one solution if $2 + 4a = 0 \Leftrightarrow a = -\frac{1}{2}$
$\tan^{-1}\left(-\frac{1}{2}\right) = -26.6°$, so the angle is $26.6°$.

2b *Alternative 1*
$x^2 + y^2 - 2x - 4y = 0 \Leftrightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 5 \Leftrightarrow (x - 1)^2 + (y - 2)^2 = 5$
Hence the centre of c_2 is $M(1{,}2)$.
The slope of radius OM is therefore $\frac{2-0}{1-0} = 2$
For the slope *r* of the tangent line we then have $2r = -1 \Leftrightarrow r = -\frac{1}{2}$
$\tan^{-1}\left(-\frac{1}{2}\right) = -26{,}6°$, so the angle is $26{,}6°$.

2b *Alternative 2*
$x^2 + y^2 - 2x - 4y = 0 \Leftrightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 5 \Leftrightarrow (x - 1)^2 + (y - 2)^2 = 5$
Hence the centre of c_2 is $M(1{,}2)$ and $\overrightarrow{OM} = \begin{pmatrix}1\\2\end{pmatrix}$.
The direction vector of the tangent line is a normal vector of $\overrightarrow{OM}$, e.g. $\begin{pmatrix}2\\-1\end{pmatrix}$
The slope of the tangent line is therefore $\frac{-1}{2} = -\frac{1}{2}$
$\tan^{-1}\left(-\frac{1}{2}\right) = -26{,}6°$, so the angle is $26{,}6°$.

2c The inner product of $\overrightarrow{OD} = \begin{pmatrix}-6\\4\end{pmatrix}$ and $\overrightarrow{OE} = \begin{pmatrix}6\\9\end{pmatrix}$ is $-6 \cdot 6 + 4 \cdot 9 = 0$

2d Triangle *ODE* has a right angle at *O*. The converse theorem of Thales then states that the centre of the circle through *O*, *D* and *E* is the midpoint of the hypothenuse *DE*. This is the point with $x = \frac{-6+6}{2} = 0$ and $y = \frac{4+9}{2} = 6\frac{1}{2}$.

2d *Alternative*
Substitute the coordinates of *O*, *D* and *E* into the formula $(x - a)^2 + (y - b)^2 = r^2$
This yields $\begin{cases} a^2 + b^2 = r^2 \\ (-6 - a)^2 + (4 - b)^2 = r^2 \\ (6 - a)^2 + (9 - b)^2 = r^2 \end{cases} \Leftrightarrow \begin{cases} a^2 + b^2 = r^2 \\ 36 + 12a + 16 - 8b = 0 \\ 36 - 12a + 81 - 18b = 0 \end{cases}$
The solutions are $a = 0$ and $b = r = 6\frac{1}{2}$.
The centre of the circle is therefore $(a, b) = \left(0{,}6\frac{1}{2}\right)$.

3a In a perforation we have $numerator = 0$ and $denominator = 0$

$denominator = 0 \Rightarrow x^2 - 4 = 0 \Leftrightarrow x^2 = 4 \Leftrightarrow x = 2 \vee x = -2$

Substition of $x = 2$ into $numerator = 0$ yields

$3 \cdot 8 - 3 \cdot 4 + 2a = 0 \Leftrightarrow 12 + 2a = 0 \Leftrightarrow 2a = -12 \Leftrightarrow a = -6$

Substitution of $x = -2$ into $numerator = 0$ yields

$3 \cdot -8 - 3 \cdot 4 - 2a = 0 \Leftrightarrow -36 - 2a = 0 \Leftrightarrow 2a = -36 \Leftrightarrow a = -18$

3b We must have $f_0(1) = g(1) = 0$ and $f_0'(1) = g'(1)$.

Substitution of $x = 1$ into $f_0(x) = \frac{3x^3-3x^2}{x^2-4}$ and into $g(x) = (1-x)\mathrm{e}^{1-x}$

indeed yields $f_0(1) = g(1) = 0$.

$$f_0'(x) = \frac{(9x^2 - 6x)(x^2 - 4) - (3x^3 - 3x^2) \cdot 2x}{(x^2 - 4)^2}$$

$$f_0'(1) = \frac{(9-6)(1-4) - (3-3) \cdot 2}{(-3)^2} = \frac{3 \cdot -3 - 0}{9} = \frac{-9}{9} = -1$$

$$g'(x) = -1 \cdot \mathrm{e}^{1-x} + (1-x)\mathrm{e}^{1-x} \cdot -1 = -\mathrm{e}^{1-x} - (1-x)\mathrm{e}^{1-x}$$

$$g'(1) = -\mathrm{e}^{1-1} - (1-1)\mathrm{e}^{1-1} = -1$$

3c $$f_0(x) = \frac{3x^3 - 3x^2}{x^2 - 4} = \frac{3x^3 - 12x - 3x^2 + 12x}{x^2 - 4} = \frac{3x^3 - 12x}{x^2 - 4} + \frac{-3x^2 + 12x}{x^2 - 4}$$

$$= \frac{3x(x^2 - 4)}{x^2 - 4} + \frac{-3 + \frac{12}{x}}{1 - \frac{4}{x^2}} = 3x + \frac{-3 + \frac{12}{x}}{1 - \frac{4}{x^2}}$$

This yields $\lim_{x \to \pm\infty} (f_0(x) - (3x - 3)) = 0$.

The oblique asymptote is therefore $y = 3x - 3$

3c *Alternative*

$$\lim_{x \to \pm\infty} f_0'(x) = \lim_{x \to \pm\infty} \frac{3x^4 - 36x^2}{x^4 - 8x^2 + 16} = 3$$

The oblique asymptote therefore has the form $y = 3x + b$.

Working $\lim_{x \to \pm\infty} \big(f_0(x) - (3x + b)\big) = 0$ yields $b = -3$ since

$$\frac{3x^3 - 3x^2}{x^2 - 4} - (3x + b) = \frac{3x^3 - 3x^2}{x^2 - 4} - \frac{(3x + b)(x^2 - 4)}{x^2 - 4} = \frac{(-3 - b)x^2 + 4b}{x^2 - 4}$$

4a $g(p) - f(p) = 2 \Leftrightarrow {}^2\log\left(\frac{16}{3-p}\right) - {}^2\log(p+2) = 2 \Leftrightarrow {}^2\log\left(\frac{16}{(3-p)(p+2)}\right) = 2$

Hence $\frac{16}{(3-p)(p+2)} = 2^2 \Leftrightarrow \frac{16}{2^2} = (3-p)(2+p) \Leftrightarrow 4 = 6 + p - p^2$

$\Leftrightarrow p^2 - p - 2 = 0 \Leftrightarrow (p-2)(p+1) = 0 \Leftrightarrow p = 2 \vee p = -1$

The equation can also be transformed into ${}^2\log\left(\frac{16}{3-p}\right) = {}^2\log(4(p+2))$ *or* ${}^2\log((3-x)(p+2)) = 2.$ *This eventually yields the same quadratic equation.*

4b The distance is minimal when $g'(p) - f'(p) = 0 \Leftrightarrow g'(p) = f'(p)$

$$g'(p) = \frac{3-p}{16} \cdot \frac{16}{(3-p)^2} \cdot \frac{1}{\ln(2)} = \frac{1}{3-p} \cdot \frac{1}{\ln(2)}$$

$g(p) = {}^2\log(16) - {}^2\log(3-x)$ *yields the same result.*

$$f'(p) = \frac{1}{p+2} \cdot \frac{1}{ln(2)}$$

$g'(p) = f'(p) \Leftrightarrow 3 - p = p + 2 \Leftrightarrow -2p = -1 \Leftrightarrow p = \frac{1}{2}$

4c $$h(x) = \frac{{}^2\log((x+2)^2)}{{}^2\log(4)} = \frac{2 \cdot {}^2\log(x+2)}{2} = {}^2\log(x+2)$$

$$or:\ f(x) = \frac{{}^4\log(x+2)}{{}^4\log(2)} = \frac{{}^4\log(x+2)}{\frac{1}{2}} = 2 \cdot {}^4\log(x+2) = {}^4\log((x+2)^2) = h(x)$$

This hold for all x in the domain of f, so for $x + 2 > 0 \Leftrightarrow x > -2$ only

5a We must have $f_a(1) = g_a(1) = 1$ and $f_a'(1) \cdot g_a'(1) = -1.$

$f_a(1) = \exp\left(\frac{1-1}{a}\right) = \exp\left(\frac{0}{a}\right) = \mathrm{e}^0 = 1;\quad g_a(1) = \exp(1 - 1^a) = \exp(1-1) = \mathrm{e}^0 = 1$

$f_a'(x) = \exp\left(\frac{x-1}{a}\right) \cdot \frac{1}{a} \Rightarrow f_a'(1) = 1 \cdot \frac{1}{a} = \frac{1}{a}$

$g_a'(x) = \exp(1 - x^a) \cdot -a \cdot x^{a-1} \Rightarrow g_a'(1) = 1 \cdot -a \cdot 1 = -a$

This yields $f_a'(1) \cdot g_a'(1) = \frac{1}{a} \cdot -a = -1$

5b $$Volume\ S_p = \pi \int_1^p \left(f_4(x)\right)^2 \mathrm{d}x = \pi \int_1^p \left(\exp\left(\frac{x-1}{4}\right)\right)^2 \mathrm{d}x = \pi \int_1^p \exp\left(\frac{x-1}{2}\right) \mathrm{d}x$$

[*since* $(\exp(X))^2 = (\mathrm{e}^X)^2 = \mathrm{e}^{2X} = \exp(2X)$]

$$\ldots = \pi \cdot \left[2\exp\left(\frac{x-1}{2}\right)\right]_1^p = \pi \cdot \left(2\exp\left(\frac{p-1}{2}\right) - 2\exp(0)\right) = 2\pi \cdot (\exp\left(\frac{p-1}{2}\right) - 1)$$

This must be equal to 2π, so we get:

$$\exp\left(\frac{p-1}{2}\right) - 1 = 1 \Leftrightarrow \exp\left(\frac{p-1}{2}\right) = 2 \Leftrightarrow \frac{p-1}{2} = \ln(2) \Leftrightarrow p = 1 + 2\ln(2)$$

6a $f(x)=0 \Leftrightarrow 2\cos^2(x)+\cos(x)-1=0 \Leftrightarrow 2y^2+y-1=0$ with $y=\cos(x)$

This yields $y=\frac{-1+\sqrt{1-4\cdot2\cdot(-1)}}{2\cdot2}=\frac{-1+3}{4}=\frac{1}{2}$ of $y=\frac{-1-\sqrt{1-4\cdot2\cdot(-1)}}{2\cdot2}=\frac{-1-3}{4}=-1$

Solutions on the interval $0\le x\le 2\pi$:

$\cos(x)=\frac{1}{2}$ for $x=\frac{1}{3}\pi$ and $x=1\frac{2}{3}\pi$

$\cos(x)=-1$ for $x=\pi$

6b $G'(x)=\frac{1}{2}+\frac{1}{4}\cdot\cos(2x)\cdot 2=\frac{1}{2}+\frac{1}{2}\cos(2x)=\frac{1}{2}+\frac{1}{2}(2\cos^2(x)-1)=\frac{1}{2}+\cos^2(x)-\frac{1}{2}$

6b Alternative 1

$G(x)=\frac{1}{2}x+\frac{1}{4}\cdot 2\sin(x)\cos(x)=\frac{1}{2}x+\frac{1}{2}\sin(x)\cos(x)$ yields

$G'(x)=\frac{1}{2}+\frac{1}{2}\cos^2(x)-\frac{1}{2}\sin^2(x)=\frac{1}{2}(\cos^2(x)+\sin^2(x))+\frac{1}{2}\cos^2(x)-\frac{1}{2}\sin^2(x)$

6b Alternative 2

$\cos(2x)=2\cos^2(x)-1$ yields $g(x)=\cos^2(x)=\frac{1}{2}+\frac{1}{2}\cos(2x)$

$G(x)$ is indeed an antiderivative of this function.

6c $f(x)=2g(x)+\cos(x)-1$

Therefore, an antiderivative is

$2G(x)+\sin(x)-x=2\left(\frac{1}{2}x+\frac{1}{4}\sin(2x)\right)+\sin(x)-x=\frac{1}{2}\sin(2x)+\sin(x)$

This yields $\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x=\left[\frac{1}{2}\sin(2x)+\sin(x)\right]_0^{\frac{1}{2}\pi}=0+1-0-0=1$

6d $h(x)=k(x) \Leftrightarrow 5x=\frac{1}{4}\pi-5x+k\cdot2\pi \Leftrightarrow 10x=\frac{1}{4}\pi+k\cdot2\pi \Leftrightarrow x=\frac{1}{40}\pi+k\cdot\frac{1}{5}\pi$

The period of the solutions is therefore $\frac{1}{5}\pi$.

This means that there are 5 solutions on the interval $0\le x\le\pi$

These are $\frac{1}{40}\pi$; $\frac{9}{40}\pi$; $\frac{17}{40}\pi$; $\frac{25}{40}\pi$ *and* $\frac{33}{40}\pi$

Stating the solutions without counting them counts as a mistake!

Note In 6a and 6c we can also first rewrite the formula of $f(x)$ using

$\cos(2x)=2\cos^2(x)-1 \Leftrightarrow 2\cos^2(x)=1+\cos(2x)$

This yields $f(x)=\cos(2x)+\cos(x)$

